

# INDIVIDUAL ERGODIC THEOREMS IN NONCOMMUTATIVE ORLICZ SPACES

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**ABSTRACT.** For a noncommutative Orlicz space associated with a semifinite von Neumann algebra, a faithful normal semifinite trace and an Orlicz function satisfying  $(\delta_2, \Delta_2)$ -condition, an individual ergodic theorem is proved.

## 1. INTRODUCTION

Development of the theory of noncommutative integration with respect to a faithful normal semifinite trace  $\tau$  defined on a semifinite von Neumann algebra  $\mathcal{M}$ , has given rise to a systematic study of various classes of noncommutative rearrangement invariant Banach spaces. The noncommutative  $L^p$ -spaces  $L^p(\mathcal{M}, \tau)$  [20, 22, 19] and, more generally, noncommutative Orlicz spaces  $L^\Phi(\mathcal{M}, \tau)$  [16, 17, 13] are important examples of such spaces.

Since every  $L^\Phi(\mathcal{M}, \tau)$  is an exact interpolation space for the Banach couple  $(L^1(\mathcal{M}, \tau), \mathcal{M})$ , for any linear operator  $T : L^1(\mathcal{M}, \tau) + \mathcal{M} \rightarrow L^1(\mathcal{M}, \tau) + \mathcal{M}$  such that

$$\|T(x)\|_\infty \leq \|x\|_\infty \quad \forall x \in \mathcal{M} \quad \text{and} \quad \|T(x)\|_1 \leq \|x\|_1 \quad \forall x \in L^1(\mathcal{M}, \tau)$$

(such operators are called Dunford-Schwartz operators), we have

$$T(L^\Phi) \subset L^\Phi \quad \text{and} \quad \|T\|_{L^\Phi \rightarrow L^\Phi} \leq 1.$$

Thus, it is natural to study noncommutative Dunford-Schwartz ergodic theorem in  $L^\Phi(\mathcal{M}, \tau)$ . The first result in this direction was obtained in [23] for the space  $L^1(\mathcal{M}, \tau)$  (as it is noticed in [3, Proposition 1.1], the class of operators  $\alpha$  that was employed in [23] coincides with the class of positive Dunford-Schwartz operators). In [10], the result of [23] was extended to the noncommutative  $L^p$ -spaces with  $1 < p < \infty$ . For general noncommutative fully symmetric spaces with non trivial Boyd indexes, an individual ergodic theorem was established in [3].

Note that the class of Orlicz spaces  $L^\Phi(\mathcal{M}, \tau)$  is significantly wider than the class of spaces  $L^p(\mathcal{M}, \tau)$ . Besides, there are Orlicz spaces  $L^\Phi(\mathcal{M}, \tau)$ , with the Orlicz function satisfying the so-called  $(\delta_2, \Delta_2)$ -condition, which have trivial Boyd index  $p_{L^\Phi} = 1$  (see Remark 2.3 below). Therefore an individual ergodic theorem for positive Dunford-Schwartz operators in Orlicz spaces does not follow from the results mentioned above.

The aim of this article is to establish an individual ergodic theorem for a positive Dunford-Schwartz operator in a noncommutative Orlicz space  $L^\Phi(\mathcal{M}, \tau)$  associated

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with an Orlicz function  $\Phi$  satisfying  $(\delta_2, \Delta_2)$ -condition. Our argument is essentially based on the notion of uniform equicontinuity in measure at zero of a sequence of linear maps from a normed space into the space of measurable operators affiliated with  $(\mathcal{M}, \tau)$ . This notion was introduced in [2] and then applied in [15] to provide a simplified proof of noncommutative individual ergodic theorem for positive Dunford-Schwartz operators in  $L^p(\mathcal{M}, \tau)$ ,  $1 < p < \infty$ .

## 2. PRELIMINARIES

Assume that  $\mathcal{M}$  is a semifinite von Neumann algebra with a faithful normal semifinite trace  $\tau$ , and let  $\mathcal{P}(\mathcal{M})$  be the complete lattice of projections in  $\mathcal{M}$ . If  $\mathbf{1}$  is the multiplicative identity of  $\mathcal{M}$  and  $e \in \mathcal{P}(\mathcal{M})$ , we denote  $e^\perp = \mathbf{1} - e$ . Let  $L^0 = L^0(\mathcal{M}, \tau)$  be the  $*$ -algebra of  $\tau$ -measurable operators. Recall that  $L^0$  is a metrizable topological  $*$ -algebra with respect to the *measure topology* that can be equivalently (see [1, Theorem 2.2]) defined by either of the families

$$V(\epsilon, \delta) = \{x \in L^0 : \|xe\|_\infty \leq \delta \text{ for some } e \in \mathcal{P}(\mathcal{M}) \text{ with } \tau(e^\perp) \leq \epsilon\}$$

or

$$W(\epsilon, \delta) = \{x \in L^0 : \|exe\|_\infty \leq \delta \text{ for some } e \in \mathcal{P}(\mathcal{M}) \text{ with } \tau(e^\perp) \leq \epsilon\},$$

$\epsilon > 0$ ,  $\delta > 0$ , of neighborhoods of zero [18].

For a positive operator  $x = \int_0^\infty \lambda de_\lambda \in L^0$  one can define

$$\tau(x) = \sup_n \tau \left( \int_0^n \lambda de_\lambda \right) = \int_0^\infty \lambda d\tau(e_\lambda).$$

If  $1 \leq p < \infty$ , then the noncommutative  $L^p$ -space associated with  $(\mathcal{M}, \tau)$  is defined as

$$L^p = (L^p(\mathcal{M}, \tau), \|\cdot\|_p) = \{x \in L^0 : \|x\|_p = (\tau(|x|^p))^{1/p} < \infty\},$$

where  $|x| = (x^*x)^{1/2}$  is the absolute value of  $x$ ; naturally,  $L^\infty = \mathcal{M}$ .

For detailed accounts on noncommutative  $L^p$ -spaces, see [19, 22].

Given  $x \in L^0$ , let  $\{e_\lambda\}_{\lambda \geq 0}$  be the spectral family of projections of  $|x|$ . If  $t > 0$ , the  $t$ -th *generalized singular number* of  $x$  [9] is defined as

$$\mu_t(x) = \inf\{\lambda > 0 : \tau(e_\lambda^\perp) \leq t\}.$$

A Banach space  $(E, \|\cdot\|_E) \subset L^0$  is called *fully symmetric* if the conditions

$$x \in E, y \in L^0, \int_0^s \mu_t(y) dt \leq \int_0^s \mu_t(x) dt \quad \forall s > 0$$

imply that  $y \in E$  and  $\|y\|_E \leq \|x\|_E$ .

If  $L \subset L^0$ , the set of all positive operators in  $L$  will be denoted by  $L_+$ .

A fully symmetric space  $(E, \|\cdot\|_E)$  is said to possess *Fatou property* if the conditions

$$x_\alpha \in E_+, x_\alpha \leq x_\beta \text{ for } \alpha \leq \beta, \text{ and } \sup_\alpha \|x_\alpha\|_E < \infty$$

imply that there exists  $x = \sup_\alpha x_\alpha \in E$  and  $\|x\|_E = \sup_\alpha \|x_\alpha\|_E$ .

Let  $m$  be Lebesgue measure on the interval  $(0, \infty)$ , and let  $L^0(0, \infty)$  be the linear space of all (equivalence classes of) almost everywhere finite complex-valued  $m$ -measurable functions on  $(0, \infty)$ . We identify  $L^\infty(0, \infty)$  with the commutative von Neumann algebra acting on the Hilbert space  $L^2(0, \infty)$  via multiplication by

the elements from  $L^\infty(0, \infty)$  with the trace given by the integration with respect to Lebesgue measure. A fully symmetric space  $E \subset L^0(\mathcal{M}, \tau)$ , where  $\mathcal{M} = L^\infty(0, \infty)$  and  $\tau$  is given by the Lebesgue integral, is called *fully symmetric function space* on  $(0, \infty)$ .

Let  $E = (E(0, \infty), \|\cdot\|_E)$  be a fully symmetric function space. For each  $s > 0$  let  $D_s : E(0, \infty) \rightarrow E(0, \infty)$  be the bounded linear operator given by

$$D_s(f)(t) = f(t/s), \quad t > 0.$$

The *Boyd indices*  $p_E$  and  $q_E$  are defined as

$$p_E = \lim_{s \rightarrow \infty} \frac{\log s}{\log \|D_s\|_E}, \quad q_E = \lim_{s \rightarrow +0} \frac{\log s}{\log \|D_s\|_E}.$$

It is known that  $1 \leq p_E \leq q_E \leq \infty$  [14, II, Ch.2, Proposition 2.b.2]. A fully symmetric function space is said to have *non-trivial Boyd indices* if  $1 < p_E$  and  $q_E < \infty$ . For example, the spaces  $L^p(0, \infty)$ ,  $1 < p < \infty$ , have non-trivial Boyd indices:

$$p_{L^p(0, \infty)} = q_{L^p(0, \infty)} = p$$

[14, II, Ch.2, 2.b.1].

If  $E$  is a fully symmetric function space on  $(0, \infty)$ , define

$$E(\mathcal{M}) = E(\mathcal{M}, \tau) = \{x \in L^0(\mathcal{M}, \tau) : \mu_t(x) \in E\}$$

and set

$$\|x\|_{E(\mathcal{M})} = \|\mu_t(x)\|_E, \quad x \in E(\mathcal{M}).$$

It is shown in [4] that  $(E(\mathcal{M}), \|\cdot\|_{E(\mathcal{M})})$  is a fully symmetric space.

If  $1 \leq p < \infty$  and  $E = L^p(0, \infty)$ , the space  $(E(\mathcal{M}), \|\cdot\|_{E(\mathcal{M})})$  coincides with the noncommutative  $L^p$ -space  $(L^p(\mathcal{M}, \tau), \|\cdot\|_p)$  because

$$\|x\|_p = \left( \int_0^\infty \mu_t^p(x) dt \right)^{1/p} = \|x\|_{L^p(\mathcal{M}, \tau)}.$$

[22, Proposition 2.4].

Since for a fully symmetric function space  $E$  on  $(0, \infty)$ ,

$$L^1(0, \infty) \cap L^\infty(0, \infty) \subset E \subset L^1(0, \infty) + L^\infty(0, \infty)$$

with continuous embeddings [12, Ch.II, §4, Theorem 4.1], we also have

$$L^1(\mathcal{M}, \tau) \cap \mathcal{M} \subset E(\mathcal{M}, \tau) \subset L^1(\mathcal{M}, \tau) + \mathcal{M},$$

with continuous embeddings.

**Definition 2.1.** A convex continuous at 0 function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  such that  $\Phi(0) = 0$  and  $\Phi(u) > 0$  if  $u \neq 0$  is called an *Orlicz function*.

**Remark 2.1.** (1) Since an Orlicz function is convex and continuous at 0, it is necessarily continuous on  $[0, \infty)$ .

(2) If  $\Phi$  is an Orlicz function, then  $\Phi(\lambda u) \leq \lambda \Phi(u)$  for all  $\lambda \in [0, 1]$ . Therefore  $\Phi$  is increasing, that is,  $\Phi(u_1) < \Phi(u_2)$  whenever  $0 \leq u_1 < u_2$ .

We will need the following lemma.

**Lemma 2.1.** *Let  $\Phi$  be an Orlicz function. Then for any given  $\delta > 0$  there exists  $t > 0$  satisfying the condition*

$$t \cdot \Phi(u) \geq u \quad \text{whenever } u \geq \delta.$$

*In particular,  $\lim_{u \rightarrow \infty} \Phi(u) = \infty$ .*

*Proof.* Since  $\Phi(u) > 0$  as  $u > 0$ , it is possible to find  $a > 0$  such that the equation  $\Phi(u) = au$  has a solution  $u = u_0 > 0$ . Then, as  $\Phi$  is convex, we have  $\Phi(u) \geq au$  for all  $u \geq u_0$ .

Fix  $\delta > 0$ . If  $\delta \geq u_0$ , then we have

$$\frac{1}{a} \cdot \Phi(u) \geq u \quad \forall u \geq \delta.$$

If  $\delta < u_0$ , then, since  $\Phi(\delta) > 0$  and  $\Phi$  is increasing on the interval  $[\delta, u_0]$ , there exists such  $s > 1$  that  $s \cdot \Phi(u) \geq au$ , or

$$\frac{s}{a} \cdot \Phi(u) \geq u, \quad \forall u \geq \delta.$$

□

**Remark 2.2.** Since an Orlicz function  $\Phi$  is continuous, increasing and such that  $\lim_{u \rightarrow \infty} \Phi(u) = \infty$ , there exists continuous increasing inverse function  $\Phi^{-1}$  from  $[0, \infty)$  onto  $[0, \infty)$ .

If  $\Phi$  is an Orlicz function,  $x \in L_+^0$  and  $x = \int_0^\infty \lambda de_\lambda$  its spectral decomposition, one can define  $\Phi(x) = \int_0^\infty \Phi(\lambda) de_\lambda$ . The *noncommutative Orlicz space* associated with  $(\mathcal{M}, \tau)$  for an Orlicz function  $\Phi$  is the set

$$L^\Phi = L^\Phi(\mathcal{M}, \tau) = \left\{ x \in L^0(\mathcal{M}, \tau) : \tau \left( \Phi \left( \frac{|x|}{a} \right) \right) < \infty \text{ for some } a > 0 \right\}.$$

The *Luxemburg norm* of an operator  $x \in L^\Phi$  is defined as

$$\|x\|_\Phi = \inf \left\{ a > 0 : \tau \left( \Phi \left( \frac{|x|}{a} \right) \right) \leq 1 \right\}.$$

**Theorem 2.1.** [13, Proposition 2.5].  $(L^\Phi, \|\cdot\|_\Phi)$  is a Banach space.

**Proposition 2.1.** *If  $x \in L^\Phi$ , then  $\Phi(|x|) \in L^0$  and  $\mu_t(\Phi(|x|)) = \Phi(\mu_t(x))$ ,  $t > 0$ . In addition,  $\tau(\Phi(|x|)) = \int_0^\infty \Phi(\mu_t(x)) dt$ .*

*Proof.* As  $x \in L^\Phi$ , we have  $\tau \left( \Phi \left( \frac{|x|}{a} \right) \right) < \infty$  for some  $a > 0$ . This implies that  $\Phi \left( \frac{|x|}{a} \right) \in L^1$ , so  $\tau \left( \left\{ \Phi \left( \frac{|x|}{a} \right) > \lambda \right\} \right) < \infty$  for all  $\lambda > 0$ . Since

$$\left\{ \Phi \left( \frac{|x|}{a} \right) > \lambda \right\} = \left\{ \Phi^{-1} \left( \Phi \left( \frac{|x|}{a} \right) \right) > \Phi^{-1}(\lambda) \right\} = \{|x| > a\Phi^{-1}(\lambda)\},$$

it follows that  $\tau(\{\Phi(|x|) > \mu\}) = \tau(\{|x| > \Phi^{-1}(\mu)\}) < \infty$  for all  $\mu > 0$ , thus  $\Phi(|x|) \in L^0$ .

By [9, Lemma 2.5, Corollary 2.8], given  $x \in L^0$ , we have  $\mu_t(\varphi(|x|)) = \varphi(\mu_t(x))$ ,  $t > 0$ , for every continuous increasing function  $\varphi : [0, \infty) \rightarrow [0, \infty)$  with  $\varphi(0) = 0$  and, in addition,  $\tau(\varphi(|x|)) = \int_0^\infty \varphi(\mu_t(x)) dt$ . Therefore  $\mu_t(\Phi(|x|)) = \Phi(\mu_t(x))$  and  $\tau(\Phi(|x|)) = \int_0^\infty \Phi(\mu_t(x)) dt$ . □

Next result follows immediately from Proposition 2.1.

**Corollary 2.1.**  $L^\Phi = \{x \in L^0 : \mu_t(x) \in L^\Phi(0, \infty)\}$  and  $\|x\|_\Phi = \|\mu_t(x)\|_\Phi$  for all  $x \in L^\Phi$ .

If  $(L^\Phi(0, \infty), \|\cdot\|_\Phi)$  is the Orlicz function space on  $(0, \infty)$  for an Orlicz function  $\Phi$ , then, by [8, Ch.2, Proposition 2.1.12], it is a rearrangement invariant function space. Since  $(L^\Phi(0, \infty), \|\cdot\|_\Phi)$  has the Fatou property [8, Ch.2, Theorem 2.1.11], Corollary 2.1, [6, Theorem 4.1], and [4, Theorem 3.4] yield the following.

**Corollary 2.2.**  $(L^\Phi, \|\cdot\|_\Phi)$  is a fully symmetric space with the Fatou property and an exact interpolation space for the Banach couple  $(L^1, \mathcal{M})$ .

We will also need the following property of the Luxemburg norm.

**Proposition 2.2.** If  $x \in L_\Phi$  and  $\|x\|_\Phi \leq 1$ , then  $\tau(\Phi(|x|)) \leq \|x\|_\Phi$ .

*Proof.* By [8, Ch.2, Proposition 2.1.10],  $\int_0^\infty \Phi(|f|)dt \leq \|f\|_\Phi$  for  $f \in L^\Phi(0, \infty)$  with  $\|f\|_\Phi \leq 1$ . Thus the result follows from Proposition 2.1 and Corollary 2.1.  $\square$

**Definition 2.2.** An Orlicz function  $\Phi$  is said to satisfy  $\Delta_2$ -condition ( $\delta_2$ -condition) if there exist  $k > 0$  and  $u_0 \geq 0$  such that

$$\Phi(2u) \leq k\Phi(u) \quad \forall u \geq u_0 \quad (\text{respectively, } \Phi(2u) \leq k\Phi(u) \quad \forall u \in (0, u_0]).$$

If an Orlicz function  $\Phi$  satisfies  $\Delta_2$ -condition and  $\delta_2$ -condition simultaneously, we will say that  $\Phi$  satisfies  $(\delta_2, \Delta_2)$ -condition. In this case  $\Phi(2u) \leq c\Phi(u)$  for all  $u \geq 0$  and some  $c > 0$ . Clearly, every space  $L^p$ ,  $1 \leq p < \infty$ , is the Orlicz space for the function  $\Phi(u) = \frac{u^p}{p}$ ,  $u \geq 0$ , which satisfies  $(\delta_2, \Delta_2)$ -condition.

**Remark 2.3.** (i) If an Orlicz function  $\Phi$  satisfies  $\Delta_2$ -condition, then the Boyd index  $q_{L^\Phi(0, \infty)} < \infty$ , that is, it is non-trivial (see [14, II, Ch.2, Proposition 2.b.5]). (ii) The function  $\Phi_\alpha(u) = u \ln^\alpha(e + u)$ ,  $\alpha \geq 0$ , is an Orlicz function that satisfies  $(\delta_2, \Delta_2)$ -condition for which the Boyd index  $p_{L^\Phi(0, \infty)}$  is trivial, that is,  $p_{L^\Phi(0, \infty)} = 1$  [21, §5].

A Banach space  $(E, \|\cdot\|_E) \subset L^0$  is said to have *order continuous norm* if  $\|x_\alpha\|_E \downarrow 0$  for every net  $\{x_\alpha\} \subset E$  with  $x_\alpha \downarrow 0$ .

**Proposition 2.3.** Let an Orlicz function  $\Phi$  satisfy  $(\delta_2, \Delta_2)$ -condition. Then

- (i) The fully symmetric space  $(L^\Phi, \|\cdot\|_\Phi)$  has order continuous norm.
- (ii) The linear subspace  $L^1 \cap \mathcal{M}$  is dense in  $(L^\Phi, \|\cdot\|_\Phi)$ .

*Proof.* (i) As shown in [8, Ch.2, §2.1], the fully symmetric space  $(L^\Phi(0, \infty), \|\cdot\|_\Phi)$  has order continuous norm. Therefore, by [5, Proposition 3.6], the noncommutative fully symmetric space  $(L^\Phi, \|\cdot\|_\Phi)$  also has order continuous norm.

(ii) Let  $x \in L_+^\Phi$ ,  $n = 1, 2, \dots$ , and  $e_n$  the spectral projection corresponding to the interval  $(n^{-1}, n)$ . It is clear that  $\{xe_n\} \subset \mathcal{M}$  and  $e_n^\perp \downarrow 0$ . Also, by (i) and [7, Theorem 3.1], we have

$$\|x - xe_n\|_\Phi = \|xe_n^\perp\|_\Phi \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now, since  $\tau(\{x > \epsilon\}) < \infty$  for all  $\epsilon > 0$  (see proof of Proposition 2.1), it follows that  $\{xe_n\} \subset L^1$ .

Since, for an arbitrary  $x \in L^\Phi$ , we have  $x = x_1 - x_2 + i(x_3 - x_4)$ , where  $x_i \in L_+^\Phi$ ,  $i = 1, \dots, 4$ , the assertion follows.  $\square$

## 3. MAIN RESULTS

Let  $\mathcal{M}$  be a semifinite von Neumann algebra with a faithful normal semifinite trace  $\tau$ ,  $L^0 = L^0(\mathcal{M}, \tau)$  the  $*$ -algebra of  $\tau$ -measurable operators affiliated with  $\mathcal{M}$ ,  $L^p = L^p(\mathcal{M}, \tau)$ ,  $1 \leq p \leq \infty$ , the noncommutative  $L^p$ -space associated with  $(\mathcal{M}, \tau)$ .

**Definition 3.1.** Let  $(X, \|\cdot\|)$  be a normed space, and let  $Y \subset X$  be such that the neutral element of  $X$  is an accumulation point of  $Y$ . A family of maps  $A_\alpha : X \rightarrow L^0$ ,  $\alpha \in I$ , is called *uniformly equicontinuous in measure (u.e.m.)* (*bilaterally uniformly equicontinuous in measure (b.u.e.m.)*) at zero on  $Y$  if for every  $\epsilon > 0$  and  $\delta > 0$  there is  $\gamma > 0$  such that, given  $x \in Y$  with  $\|x\| < \gamma$ , there exists  $e \in \mathcal{P}(\mathcal{M})$  such that

$$\tau(e^\perp) \leq \epsilon \quad \text{and} \quad \sup_{\alpha \in I} \|A_\alpha(x)e\|_\infty \leq \delta \quad (\text{respectively, } \sup_{\alpha \in I} \|eA_\alpha(x)e\|_\infty \leq \delta).$$

**Remark 3.1.** As explained in [15, Introduction], in the commutative case, the notion of uniform equicontinuity in measure at zero of a family  $\{A_n\}_{n \in \mathbb{N}}$  coincides with the continuity in measure at zero of the maximal operator associated with this family.

**Definition 3.2.** A sequence  $\{x_n\} \subset L^0$  is said to converge to  $x \in L^0$  *almost uniformly (a.u.)* (*bilaterally almost uniformly (b.a.u.)*) if for every  $\epsilon > 0$  there exists such a projection  $e \in \mathcal{P}(\mathcal{M})$  that  $\tau(e^\perp) \leq \epsilon$  and  $\|(x - x_n)e\|_\infty \rightarrow 0$  (respectively,  $\|e(x - x_n)e\|_\infty \rightarrow 0$ ).

A proof of the following fact can be found in [15, Theorem 2.1].

**Proposition 3.1.** Let  $(X, \|\cdot\|)$  be a Banach space,  $A_n : X \rightarrow L^0$  a sequence of additive maps. If the family  $\{A_n\}$  is u.e.m. (b.u.e.m.) at zero on  $X$ , then the set

$$\{x \in X : \{A_n(x)\} \text{ converges a.u. (respectively, b.a.u.)}\}$$

is closed in  $X$ .

**Definition 3.3.** A linear map  $T : L^1 + L^\infty \rightarrow L^1 + L^\infty$  such that

$$\|T(x)\|_\infty \leq \|x\|_\infty \quad \forall x \in \mathcal{M} \quad \text{and} \quad \|T(x)\|_1 \leq \|x\|_1 \quad \forall x \in L^1.$$

is called a *Dunford-Schwartz operator*.

If  $T$  is a Dunford-Schwartz operator (positive Dunford-Schwartz operator), we will write  $T \in DS$  (respectively,  $T \in DS^+$ ). If  $T \in DS$ , consider its ergodic averages

$$(1) \quad A_n(x) = A_n(T, x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k(x), \quad x \in L^1 + L^\infty.$$

Here is a noncommutative maximal ergodic inequality due to Yeadon [23] (for the assumption  $T \in DS^+$ , see a clarification given in [3, Proposition 1.1, Remark 1.2]):

**Theorem 3.1.** Let  $T \in DS^+$  and  $A_n : L^1 \rightarrow L^1$ ,  $n = 1, 2, \dots$  be given by (1). Then for every  $x \in L^1_+$  and  $\nu > 0$  there exists a projection  $e \in \mathcal{P}(\mathcal{M})$  such that

$$\tau(e^\perp) \leq \frac{\|x\|_1}{\nu} \quad \text{and} \quad \sup_n \|eA_n(x)e\|_\infty \leq \nu.$$

Now, let  $\Phi$  be an Orlicz function,  $L^\Phi = L^\Phi(\mathcal{M}, \tau)$  the corresponding noncommutative Orlicz space,  $\|\cdot\|_\Phi$  the Luxemburg norm in  $L^\Phi$ .

As  $L^\Phi$  is an exact interpolation space for the Banach couple  $(L^1, \mathcal{M})$  (see Corollary 2.2),

$$(2) \quad T(L^\Phi) \subset L^\Phi \quad \text{and} \quad \|T\|_{L^\Phi \rightarrow L^\Phi} \leq 1,$$

hold for any  $T \in DS$ , and we have the following.

**Proposition 3.2.** *If  $T \in DS^+$ , then the family  $A_n : L^\Phi \rightarrow L^\Phi$ ,  $n = 1, 2, \dots$ , given by (1) is b.u.e.m. at zero on  $(L^\Phi, \|\cdot\|_\Phi)$ .*

*Proof.* It is easy to verify (see [15, Lemma 4.1]) that it is sufficient to show that  $\{A_n\}$  is b.u.e.m. at zero on  $(L_+^\Phi, \|\cdot\|_\Phi)$ .

Fix  $\epsilon > 0, \delta > 0$ . By Lemma 2.1, there exists  $t > 0$  such that

$$t \cdot \Phi(\lambda) \geq \lambda \quad \text{as soon as} \quad \lambda \geq \frac{\delta}{2}.$$

Let  $\nu > 0$  and  $0 < \gamma \leq 1$  be such that  $\nu \leq \frac{\delta}{2t}$  and  $\frac{\gamma}{\nu} \leq \epsilon$ .

Take  $x \in L_+^\Phi$  with  $\|x\|_\Phi \leq \gamma$ , and let  $x = \int_0^\infty \lambda de_\lambda$  be its spectral decomposition. Then we can write

$$x = \int_0^{\delta/2} \lambda de_\lambda + \int_{\delta/2}^\infty \lambda de_\lambda \leq x_\delta + t \cdot \int_{\delta/2}^\infty \Phi(\lambda) de_\lambda \leq x_\delta + t \cdot \Phi(x),$$

where  $x = \int_0^{\delta/2} \lambda de_\lambda$  and  $\Phi(x) = \int_0^\infty \Phi(\lambda) de_\lambda$ .

As  $\|x_\delta\|_\infty \leq \frac{\delta}{2}$  and  $T \in DS^+$ , we have

$$\sup_n \|A_n(x_\delta)\|_\infty \leq \frac{\delta}{2}.$$

Besides, by Proposition 2.2,  $\|x\|_\Phi \leq 1$  implies that  $\|\Phi(x)\|_1 \leq \|x\|_M \leq \gamma$ . Since  $\Phi(x) \in L_+^1$ , in view of Theorem 3.1, one can find a projection  $e \in \mathcal{P}(\mathcal{M})$  such that

$$\tau(e^\perp) \leq \frac{\|\Phi(x)\|_1}{\nu} \leq \frac{\gamma}{\nu} \leq \epsilon \quad \text{and} \quad \sup_n \|eA_n(\Phi(x))e\|_\infty \leq \nu \leq \frac{\delta}{2t}.$$

Consequently,

$$\sup_n \|eA_n(x)e\|_\infty \leq \sup_n \|eA_n(x_\delta)e\|_\infty + t \cdot \sup_n \|eA_n(\Phi(x))e\|_\infty \leq \frac{\delta}{2} + t \cdot \frac{\delta}{2t} = \delta,$$

and the proof is complete.  $\square$

Here is an individual ergodic theorem for noncommutative Orlicz spaces:

**Theorem 3.2.** *Assume that an Orlicz function  $\Phi$  satisfy  $(\delta_2, \Delta_2)$ -condition. Then, given  $T \in DS^+$  and  $x \in L^\Phi$ , the averages (1) converge b.a.u. to some  $\hat{x} \in L^\Phi$ .*

*Proof.* Since, by Proposition 2.3, the set  $L^1 \cap \mathcal{M} \subset L^2$  is dense in  $L^\Phi$  and the averages (1) converge a.u., hence b.a.u., for every  $x \in L^2$  (see, for example, [15, Theorem 4.1]), it follows from Propositions 3.2 and 3.1 that for any  $x \in L^\Phi$  the averages (1) converge b.a.u. to some  $\hat{x} \in L^0$ .

It is clear that a b.a.u. convergent sequence in  $L^0$  converges in measure, hence  $A_n(x) \rightarrow \hat{x}$ ,  $x \in L^\Phi$ , in measure. Since, by Corollary 2.2,  $L^\Phi$  has the Fatou property, its unit ball is closed in the measure topology [6, Theorem 4.1], and (2), hence  $\sup_n \|A_n(x)\|_{L^\Phi \rightarrow L^\Phi} \leq \|x\|_\Phi$ , implies that  $\hat{x} \in L^\Phi$ .  $\square$

**Remark 3.2.** It was shown in [3, Theorem 5.2] that if  $E(0, \infty)$  is a fully symmetric function space with Fatou property and non-trivial Boyd indices and  $T \in DS^+$ , then for any  $x \in E(\mathcal{M}, \tau)$  the averages  $A_n(x)$  converge b.a.u. to some  $\hat{x} \in E(\mathcal{M}, \tau)$ . According to Remark 2.3 (ii), there exists an Orlicz function  $\Phi$  that satisfies  $(\delta_2, \Delta_2)$ -condition for which the Boyd index  $p_{L^\Phi(0, \infty)}$  is trivial. Thus, Theorem 3.2 does not follow from Theorem [3, Theorem 5.2].

Now we shall turn to a class of Orlicz spaces for which the averages (1) converge a.u. The following fundamental result is crucial.

**Theorem 3.3** (Kadison's inequality [11]). *If  $S : \mathcal{M} \rightarrow \mathcal{M}$  is a positive linear operator such that  $S(\mathbf{1}) \leq \mathbf{1}$ , then  $S(x)^2 \leq S(x^2)$  for every  $x^* = x \in \mathcal{M}$ .*

**Definition 3.4.** We call a convex function  $\Phi$  on  $[0, \infty)$  *2-convex* if the function  $\tilde{\Phi}(u) = \Phi(\sqrt{u})$  is also convex.

For example,  $\Phi(u) = \frac{u^p}{p}$ ,  $u \geq 0$ , is 2-convex that satisfies  $(\delta_2, \Delta_2)$ -condition whenever  $p \geq 2$ .

It is clear that if  $\Phi$  is a 2-convex Orlicz function, then  $\tilde{\Phi}$  is also an Orlicz function, and it is easy to verify the following.

**Proposition 3.3.** *If  $\Phi$  be a 2-convex Orlicz function, then  $x^2 \in L_{\tilde{\Phi}}^+$  and  $\|x^2\|_{\tilde{\Phi}} = \|x\|_{\Phi}^2$  for every  $x \in L_{\Phi}^+$ .*

**Proposition 3.4.** *Let  $\Phi$  be a 2-convex Orlicz function. Then the family  $\{A_n\}$  given by (1) is u.e.m. at zero on  $(L^\Phi, \|\cdot\|_M)$ .*

*Proof.* As it was noticed earlier, it is sufficient to show that  $\{A_n\}$  is u.e.m. at zero on  $(L_{\Phi}^+, \|\cdot\|_{\Phi})$ .

Fix  $\epsilon > 0$ ,  $\delta > 0$ . By Proposition 3.2,  $\{A_n\}$  is b.u.e.m. at zero on  $(L^{\tilde{\Phi}}, \|\cdot\|_{\tilde{\Phi}})$ . Therefore there exists  $\gamma > 0$  such that, given  $y \in L^{\tilde{\Phi}}$  with  $\|y\|_{\tilde{\Phi}} < \gamma$ ,

$$\sup_n \|eA_n(y)e\|_{\infty} \leq \delta^2 \text{ for some } e \in \mathcal{P}(\mathcal{M}) \text{ with } \tau(e^\perp) \leq \epsilon.$$

Now, let  $x \in L_{\Phi}^+$  be such that  $\|x\|_{\Phi} < \gamma^{1/2}$ . Then, due to Proposition 3.3,  $x^2 \in L^{\tilde{\Phi}}$  and  $\|x^2\|_{\tilde{\Phi}} = \|x\|_{\Phi}^2 \leq \gamma$ , implying that there is a projection  $e \in \mathcal{P}(\mathcal{M})$  such that

$$\sup_n \|eA_n(x^2)e\|_{\infty} \leq \delta^2 \text{ and } \tau(e^\perp) \leq \epsilon.$$

Then, by Kadison's inequality,

$$\begin{aligned} \left[ \sup_n \|A_n(x)e\|_{\infty} \right]^2 &= \sup_n \|A_n(x)e\|_{\infty}^2 = \sup_n \|eA_n(x)^2e\|_{\infty} \leq \\ &\leq \sup_n \|eA_n(x^2)e\|_{\infty} \leq \delta^2, \end{aligned}$$

which completes the proof.  $\square$

Now, as in Theorem 3.2, we obtain the following.

**Theorem 3.4.** *If an Orlicz function  $\Phi$  satisfies  $(\delta_2, \Delta_2)$ -condition and is 2-convex, then, given  $T \in DS^+$  and  $x \in L^\Phi$ , the averages (1) converge a.u. to some  $\hat{x} \in L^\Phi$ .*



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